FINAL: INTRODUCTION TO ALGEBRAIC GEOMETRY

Date: $4^{\textbf{th}}$ **May 2022**
The Total points is **110** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (10+5=15 points) Let $\phi : X \to Y$ be a surjective morphism of affine varieties. Show that the induced map of k-algebra $\phi^{\#} : \mathcal{O}_Y \to \mathcal{O}_X$ is injective. Also show that the converse is false.

(2) (5+15=20 points) Let X be an algebraic subset of a projective space $\mathbb{P}^n$. When is X called irreducible? Assuming X to be an algebraic subset of a projective space, show that X is irreducible iff the homogeneous coordinate ring of $X \subset \mathbb{P}^n$ is a domain.

(3) (5+10+5=20 points) Let R be a ring and I an ideal of R.
   (a) Define minimal primes of I.
   (b) Compute the irreducible components of the affine algebraic subset of $\mathbb{A}^2_{\mathbb{C}}$ defined by the polynomial $f(x,y) = (x^2 - y^2)(x^2 + y^2 - 1)(x^4 - y^4)$.
   (c) What are the minimal primes of the ideal (f) in $\mathbb{C}[x,y]$? Also Compute $\sqrt{(f)}$.

(4) (5+15=20 points) Let X be an affine variety over an algebraically closed field k. Define the function field of X. Show that there exists n such that the function field of X is a finite field extension of the field $k(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are algebraically independent.

(5) (10+10=20 points)Let $f = x^7 - y^7 + z^7$ be an irreducible homogeneous polynomial in $\mathbb{C}[x,y,z]$ and let $X = Z(f)$ be the projective variety defined by f. Find the locus where the rational function $\bar{x}/\bar{y}$ on X is regular. Find a nonconstant rational function on X which is regular away from the point $[1,1,0]$.

(6) (15 points) Which of the following formulas define a morphism between projective varieties? Give reasons.
   (a) $f : \mathbb{P}^2 \to \mathbb{P}^3$ where $f([a,b,c]) = [a + b^2, ab + c^2, abc, b + ca]$.
   (b) $f : \mathbb{P}^2 \to \mathbb{P}^3$ where $f([a,b,c]) = [a^2 b + b^3, bc^2, abc - b^2 c, b^3]$.
   (c) $f : \mathbb{P}^2 \to \mathbb{P}^3$ where $f([a,b,c]) = [a + b, b + c, c + a, a + b + c]$